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NOTE

A Best Approximation for Constant e and an Improvement to Hardy's Inequality

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In this paper, the best approximating form of constant e and Hardy's inequality are discussed, and the improved conclusions of earlier work are achieved.

Recently, Y. Bicheng [1, 2] attained the inequalities involving constant e,

$$e\left(1 - \frac{1}{2x+1}\right) < \left(1 + \frac{1}{x}\right)^x < e\left[1 - \frac{1}{2(x+1)}\right], \tag{1}$$

This is equivalent to

$$\frac{e}{2(x+1)} < e - \left(1 + \frac{1}{x}\right)^x < \frac{e}{2x+1}, \tag{2}$$

or

$$\frac{2x+2}{2x+1}\left(1 + \frac{1}{x}\right)^x < e < \frac{2x+1}{2x}\left(1 + \frac{1}{x}\right)^x. \tag{3}$$

As an application of inequalities (1), he obtained a strengthened Hardy's inequality.

The main results of this paper are presented as follows:

(I) The above inequalities involving constant e can be strengthened as $(x \geq 1)$

$$e\left(1 - \frac{7}{14x+12}\right) < \left(1 + \frac{1}{x}\right)^x < e\left(1 - \frac{6}{12x+11}\right), \tag{4}$$

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which has the equivalences

$$\frac{6e}{12x+11} < e - \left(1+\frac{1}{x}\right)^x < \frac{7e}{14x+12}, \tag{5}$$

and

$$\frac{12x+11}{12x+5}\left(1+\frac{1}{x}\right)^x < e < \frac{14x+12}{14x+5}\left(1+\frac{1}{x}\right)^x. \tag{6}$$

Strengthening this we have

$$\frac{12x+11}{12x+5}\left(1+\frac{1}{x}\right)^x < e < \frac{12x+1}{12x+7}\left(1+\frac{1}{x}\right)^{x+1} < \frac{14x+12}{14x+5}\left(1+\frac{1}{x}\right)^x. \tag{7}$$

(II) When $x \to \infty$,

(i) Using $f(x,a,b) = ((x+1+a)/(x+1+b))(1+1/x)^x$ to approximate e, then $a = -\frac{1}{12}, b = \frac{7}{12}$ are the best values.

(ii) Using $f_1(x,a_1,b_1) = ((x+1+a_1)/(x+1+b_1))(1+1/x)^{x+1}$ to approximate e, then the best values are $a_1 = -\frac{11}{12}, b_1 = -\frac{5}{12}$. In other words, the first two inequalities of inequality (7) are the best approximation for constant e which can't be strengthened any more unless the form of $f(x,a,b)$ is changed and new parameters are added (refer to Remarks).

(III) Simultaneously, Hardy's inequality is correspondingly improved: if $0 < \lambda_{n+1} \le \lambda_n, \Lambda_n = \sum_{m=1}^{n} \lambda_m, a_n \ge 0 (n \in N), 0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_{n+1}\left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty}\left[1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\right]\lambda_n a_n. \tag{8}$$

Now we will prove the above results.

1. *The Proof of Conclusion* (I). Step 1. Still denote $f(x,a,b) = ((x+1+a)/(x+1+b))(1+1/x)^x$. Setting

$$g(x,a,b) = \ln f(x,a,b) - 1 = \ln(x+1+a) + x\ln\left(1+\frac{1}{x}\right) - \ln(x+1+b) - 1,$$

it's expansion at $u = x+1$ can be expressed as

$$g(x,a,b) = \sum_{k=1}^{\infty} \frac{A_k}{u^k}, \tag{9}$$

where $A_k = ((-1)^{k-1}/k)a^k + ((-1)^k/k)a^k - 1/k(k+1)$. Then, when $a = -\frac{1}{12}$ and $b = -\frac{7}{12}$, we have $A_1 = A_2 = 0$; and if $k \geq 3$, in view of $12^k > (k+1)(7^k - 1)$, we find $A_k < 0$. Therefore

$$g\left(x, -\frac{1}{12}, -\frac{7}{12}\right) < 0,$$

namely,

$$f\left(x, -\frac{1}{12}, -\frac{7}{12}\right) = \frac{12x+11}{12x+5}\left(1 + \frac{1}{x}\right)^x < e.$$

Step 2. Analogous to Step 1, we still denote

$$f_1(x, a_1, b_1) = \frac{x+1+a_1}{x+1+b_1}\left(1 + \frac{1}{x}\right)^{x+1},$$

$$\begin{aligned} g_1(x, a_1, b_1) &= \ln f(x, a_1, b_1) - 1 = (x+1)\ln\left(1 + \frac{1}{x}\right) \\ &\quad + \ln(x+1+a_1) - \ln(x+1+b_1) - 1 \\ &= \sum_{k=1}^{\infty} \frac{B_k}{u^k}, \end{aligned}$$

where $B_k = ((-1)^{k-1}/k)a_1^k + ((-1)^k/k)b_1^k + 1/(k+1)$. Then when $a_1 = -\frac{11}{12}, b_1 = -\frac{5}{12}, B_1 = B_2 = 0$ is valid, and if $k \geq 3$, $B_k < 0$ follows. Hence $f_1(x, -\frac{11}{12}, \frac{5}{12}) < 0$, namely,

$$e < f_1\left(x, -\frac{11}{12}, -\frac{5}{12}\right) = \frac{12x+1}{12x+7}\left(1 + \frac{1}{x}\right)^{x+1}.$$

Last, if $x \geq 1$, we obtain

$$\begin{aligned} f_1\left(x, -\frac{11}{12}, -\frac{5}{12}\right) &= \frac{12x+1}{12x+7}\left(1 + \frac{1}{x}\right)^{x+1} < \frac{14x+12}{14x+5}\left(1 + \frac{1}{x}\right)^x \\ &= f\left(x, -\frac{1}{7}, -\frac{9}{14}\right). \end{aligned}$$

Thus inequality (7) is true. This completes the proof of Conclusion (I).

It also easy to find that $f(x, -\frac{11}{12}, -\frac{7}{12})$ is increasing. However, $f_1(x, -\frac{11}{12}, -\frac{5}{12})$ is decreasing, each of them being approximate to e.

2. *The Proof of Conclusion* (II). In fact, for $x \to \infty$, $g(x, a, b)$ is infinite small, hence $f(x, a, b)$ is still infinite small. If a and b are wanted to be the best values, as a matter of course, the order of $g(x, a, b)$ is better as it

gets higher. According to inequality (9), when $a = -\frac{1}{12}, b = -\frac{7}{12}, A_1 = A_2 = 0$ is valid, and $g(x, a, b)$ is infinite small, and also the highest order infinite small. Thus the degree of $f(x, a, b)$ approximating e is best.

Similarly, for $a_1 = -\frac{11}{12}, b_1 = -\frac{5}{12}, g_1(x, a_1, b_1)$ is the highest order infinite small (2 order infinite small), thereby $f_1(x, -\frac{11}{12}, -\frac{5}{12})$ is also the best approximation of e.

Remark. In accordance with the above analysis, for $a - b = \frac{1}{2}$ but $a \neq -\frac{1}{12}$, in Eq. (9), we have $A_1 = 0, A_2 \neq 0$, when $x \to \infty, g(x, a, b)$ is 1 order infinite small. Then the closer a approaches $-\frac{1}{12}$, the better $f(x, a, b)$ approximates e. In inequalities (1) of [1, 2], the values of a and b are successively $a = 0, b = -\frac{1}{2}$, and $a = -\frac{1}{2}, b = -1$.

On the other hand, when $a_0 < -\frac{1}{12}$ and $a_0 - b_0 = \frac{1}{2}$, it's certain that existing x_0, and for $x \geq x_0$,$((12x + 1)/(12x + 7))(1 + 1/x)^{x+1} < ((x + 1 + a_0)/(x + 1 + b_0))(1 + 1/x)^x$ is true. For example, if $a_0 = -\frac{1}{11}, b_0 = -\frac{13}{22}$, then $x_0 = 9$.

3. *The Proof of Conclusion* (III). Setting $0 < \lambda_{n+1} \leq \lambda_n, \Lambda_n = \sum_{m=1}^{n} \lambda_m, a_n \geq 0 (n \in N), 0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then by virtue of the proof of article [2] and inequality (4), we have

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_{n+1}\left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} &< \sum_{m=1}^{\infty} \left(1 + \frac{1}{\Lambda_m/\lambda_m}\right)^{\Lambda_m/\lambda_m} \lambda_m a_m \\ &< e \sum_{m=1}^{\infty} \left[1 - \frac{6}{12\Lambda_m/\lambda_m + 11}\right] \lambda_m a_m \\ &= e \sum_{m=1}^{\infty} \left[1 - \frac{6\lambda_m}{12\Lambda_m + 11\lambda_m}\right] \lambda_m a_m.\end{aligned}$$

4. *The Approximate Calculation of e.* In view of Eq. (2) or Eq. (5), we take $(1 + \frac{1}{x})^x$ as the approximate value of e. It is necessary that x must become very big if the degree of accuracy is to be improved. However, it's more convenient to evaluate the approximate value of e taking advantage of Eq. (7). For example,

x	$\frac{2x+2}{2x+1}\left(1+\frac{1}{x}\right)^x$	$\frac{12x+11}{12x+5}\left(1+\frac{1}{x}\right)^x$	$\frac{12x+1}{12x+7}\left(1+\frac{1}{x}\right)^{x+1}$	$\frac{2x+1}{2x}\left(1+\frac{1}{x}\right)^x$
8	2.7167	2.718207	2.718361	2.7261
9	2.7170	2.718228	2.718338	2.7245
10	2.7172	2.718242	2.718323	2.7234

Remark. (1°) Changing the form of $f(x, a, b)$ and adding new parameters, we an achieve a more accurate approximate form. For example, the author uses the form

$$f(x,a,b) = \left(1 + \frac{1}{x}\right)^{x} \left(\frac{x+a}{x+b}\right)^{\alpha}$$

to make $\ln f - 1$ to be 3 order infinite small and obtains that $a = -\frac{1}{3} + \frac{1}{\sqrt{6}}$, $b = -\frac{1}{3} - \frac{1}{\sqrt{6}}$, $\alpha = \frac{\sqrt{6}}{4}$ and $a = -\frac{1}{3} - \frac{1}{\sqrt{6}}$, $b = -\frac{1}{3} + \frac{1}{\sqrt{6}}$, $\alpha = -\frac{\sqrt{6}}{4}$. Because a, b, α are all irrational numbers, they have little value of application.

(2°) It's very analogous to prove that for $x \to \infty$, $f(x, \alpha) = (1 + \frac{1}{x})^{x+\alpha}$ approximates e with the best value $\alpha = \frac{1}{2}$. Then $\ln f(x, \alpha) - 1$ is 1 order infinite small, and $f(x, \frac{1}{2})$ decreasingly approximates e.

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